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Generalized van der Corput Sequences and Dynamical Systems

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Key words
- Van der Corput seq.
- low discrepancy
- Orbit of dynamical systems

— 0 —

§0 Intro.

$x_0, x_1, \cdots, x_N, \cdots \in [0,1)^k$

I : interval of $[0,1]^k$

$A(I, N, \{x_n\}) := {}^{\#}\{ n \mid x_n \in I,\ n < N \}$

Def 1.

$$D_N = D_N(\{x_n\}) := \sup_{I \subset T^k} \left| \frac{A(I, N, \{x_n\})}{N} - \lambda_k(I) \right|$$

is called <u>discrepancy</u> of $\{x_n\}_{n=0}^{\infty}$

star discrepancy

$$D_N^*(\{x_n\}) := \sup_{J = [0,y_1) \times \cdots \times [0,y_k)} \left| \quad \text{"} \quad \right|$$

$$D_N^* \leqq D_N \leqq 4^k D_N^*$$

Th (Koksma-Hlawka's Ineq.)

f : be of bounded variation on $[0,1]^k$ in the sense of Hardy & Krause.

then

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) - \int_{[0,1]^k} f(x)\,dx \right| \leqq \underline{V(f)}\, \underline{D_N^*(\{x_n\})}$$

— 1 —

Def 2. $\{x_n\} \in [0,1]^k$ is called a <u>low discrepancy</u> sequence if

$$\exists c: \quad D_N \leq c \log^k N / N \quad \text{for all } N$$

§1 van der Corput sequence.

$$n = a_k 2^k + a_{k-1} 2^{k-1} + \cdots + a_0 2^0 \quad , a_k \neq 0$$

$$\varphi(n) := (a_k, a_{k-1}, \cdots, a_0)$$

$$\boxed{x_n := \frac{a_0}{2} + \frac{a_1}{2^2} + \cdots + \frac{a_k}{2^{k+1}}} \quad , n = 0, 1, 2, \cdots$$

$\{x_n\}$ is called <u>Van der Corput</u> seq.

Prop 1 Van der Corput seq. is a low discrepancy seq.

List of $\varphi(n) = (a_k, \cdots a_0)$ and $x_n = \frac{a_0}{2} + \frac{a_1}{2^2} + \cdots + \frac{a_k}{2^{k+1}}$

```
phi[0]={0}  x[0]={0.}                     ] P1
phi[1]={1}  x[1]={0.5}                    ] P1  P2
phi[2]={1, 0}  x[2]={0.25}                      P2
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phi[32]={1, 0, 0, 0, 0, 0}  x[32]={0.015625}
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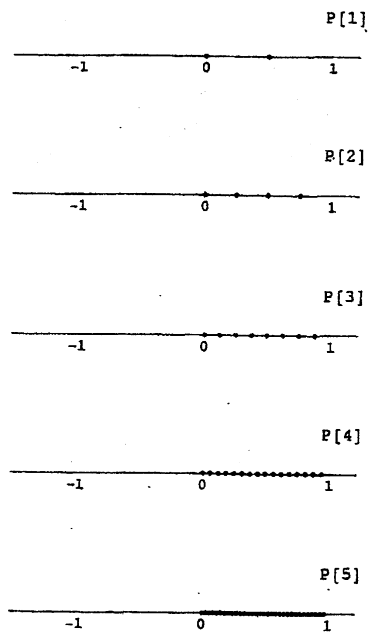


figure of P_k, (Def. of P_k found in page 5)

sketch of proof.

$$\boxed{P_k := \{x_0, x_1, \cdots x_{2^k-1}\}}$$

Lemma 1

(1) $N = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}$

$$\{\underbrace{x_0, \cdots x_{2^{k_1}-1}}_{=}\,|\,\underbrace{x_{2^{k_1}} \cdots x_{2^{k_2}-1}}_{=}\,|\,x_{2^{k_2}} \cdots \cdots \,|\,\underbrace{\cdots x_{N-1}}_{=}\}$$

$$= P_{k_1} \cup \left(P_{k_2} + \frac{1}{2^{k_1+1}}\right) \cup \cdots \cup \left(P_{k_s} + \frac{1}{2^{k_1+1}} + \cdots + \frac{1}{2^{k_{s-1}+1}}\right)$$

(2) $P_1 = \{0, 1/2\}$

$P_2 = P_1 \cup (P_1 + 1/2^2)$

$\vdots$

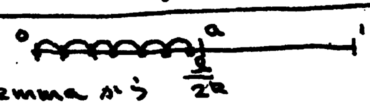
$P_k = P_{k-1} \cup (P_{k-1} + 1/2^k)$

Self-similar

Lemma 2

$$\boxed{\left|\#(P_k \cap [0,a)) - a \cdot 2^k\right| \leq 1}$$

Bounded Remainder



$\frac{l-1}{2^k} < a \leq \frac{l}{2^k}$

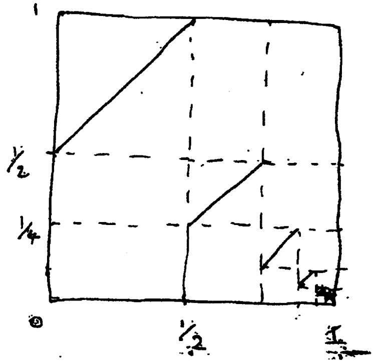
この2つの Lemma から

$$\left|\#\{x_0, \cdots, x_{N-1}\} \cap [0,a) - a \cdot N\right|$$

$$= \left|\sum_{i=1}^{s}\left(\#\left(P_{k_i} + \frac{1}{2^{k_1+1}} + \cdots + \frac{1}{2^{k_{i-1}+1}}\right) \cap [0,a) - a \cdot 2^{k_i}\right)\right|$$

$$\leq k_1 \doteqdot \log N \qquad (\because)\ 2^{k_1} \leq N < 2^{k_1+1}$$

Plot of (x_n, x_{n+1}), $n=0,1,2,\dots$



Then we have the graph of
<u>Kakutani transf.</u> T_K
(adding machine)

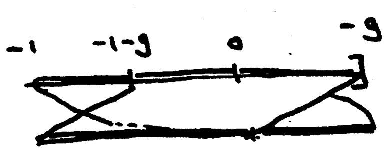
Prop $T_K : [0,1) \to [0,1)$ Kakutani tr.
then $\boxed{x_n = T_K^n(0)}$

§2 rotations

$$g := \frac{1-\sqrt{5}}{2}$$

$$T_g : (-1,-g] \longrightarrow (-1,-g]$$

$$T_g(x) = \begin{cases} x+g & \text{if } x \in (-1-g,-g] \\ x+1 & \text{if } x \in (-1,-1+g] \end{cases}$$



Proposition $x_n := T_g^n(0)$, $n=0,1,2,\dots$
is a <u>low discrepancy seq.</u>

Sketch of the proof (new!?)
<u>Fibonacci substitution</u>

$$\sigma : 1 \to 12$$
$$2 \to 1$$

$$L_\sigma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad L_\sigma^n = \begin{bmatrix} g_n & g_{n-1} \\ p_n & p_{n-1} \end{bmatrix}$$

$\sigma(1) = 12$ $\quad |\sigma^0(1)| = 1$

$\quad\quad\quad\quad\quad\quad\;\; |\sigma(1)| = 2$

$\sigma^2(1) = 121$ $\quad |\sigma^2(1)| = 3$

$\sigma^3(1) = 12112$ $\quad |\sigma^3(1)| = 5$

$\vdots \qquad\qquad \vdots$

$\sigma^n(1) = s_1 s_2 \cdots s_{l_n}$ $\quad |\sigma^n(1)| = l_n$

$\boxed{l_n = l_{n-1} + l_{n-2}}$, $l_0 = 1$, $l_1 = 2$

$$f : \bigcup \{1,2\}^n \longrightarrow \mathbb{Z}^2$$

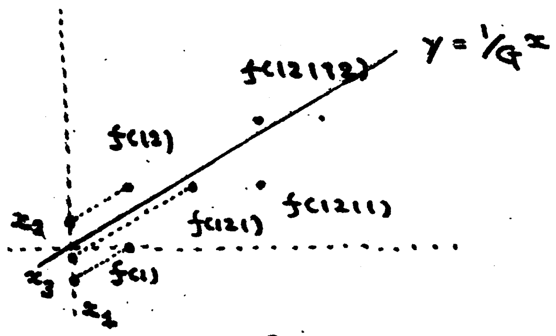
$$1 \longmapsto f(1) = e_1$$

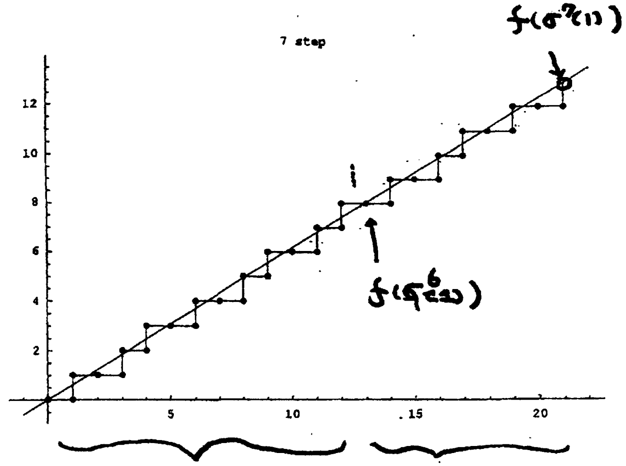
$$2 \longmapsto f(2) = e_2$$

$$f(s_1 \cdots s_k) := f(s_1) + \cdots + f(s_k)$$

$$\hat{\pi} : \mathbb{R}^2 \to y\text{-axis along } \begin{bmatrix} 1 \\ 1/\varphi \end{bmatrix}$$

$$\boxed{x_n := \hat{\pi} f(s_1 \cdots s_n) \quad (x_0 = 0)}$$





$f(s_1 \cdots s_k)$, $0 \le k < l_6$

$f(s_1 \cdots s_k)$, $l_6 \le k < l_7$

$= f(s_1 \cdots s_{l_6}) + f(s_1 \cdots s_k)$, $0 \le k < l_5$

figure of $f(s_1, \cdots, s_k)$ $\;0 \le k < l_7$

proposition $\quad x_n := \hat{\pi} f(s_1 \cdots s_n)$ satisfies

(1) $x_n = T_g^n(0)$

(2) $x_n = \sum_{i=0}^{k} a_i g^{i+1}$

where $(a_0, a_1, \cdots, a_k)$ is given by

$n = a_k l_k + a_{k-1} l_{k-1} + \cdots + a_0 l_0$

In other words,

$\{l_n\}$ given	$\{2^n\}$ given
$\forall n = a_k l_k + a_{k-1} l_{k-1} + \cdots + a_0 l_0$	$n = a_k 2^k + a_{k-1} 2^{k-1} + \cdots + a_0 2^0$
$a_k \in \{0, 1\}$	$a_i \in \{0, 1\}$
$(a_j, a_{j-1}) \neq (1, 1)$	

$\Downarrow$

$x_n := \sum_{i=0}^{k} a_i g^{i+1} = \sum_{i=0}^{k} a_i \left(\frac{-1 \pm \sqrt{5}}{2}\right)^{i+1}$ $\qquad x_n := \sum_{i=0}^{k} \frac{a_i}{2^{i+1}}$

This is a kind of van der Corput seq.

$G^2 = G + 1$ $\quad$ solution of $x^2 - x - 1 = 0$, $x = \frac{1 \pm \sqrt{5}}{2}$ $\quad G' = g$


```
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phi[33]={1, 0, 1, 0, 1, 0, 1}  x[33]={-0.978714}
```

List of

$n = a_k l_k + a_{k-1} l_{k-1} + \cdots + a_0 l_0$

$\varphi(n) = (a_k, a_{k-1}, \cdots, a_0)$

and

$x_n = \sum_{i=0}^{k} a_i g^{i+1}$

property of $P_k = \{x_0, x_1, \cdots, x_{g_k - 1}\}$

(1) $P_{k+1} = P_k \cup (P_{k-1} + 1/g_k)$ self-similar

(2) $|\#(P_k \cap [0,a)) - \lambda([0,a))| \leqq 1$ Bounded Remainder

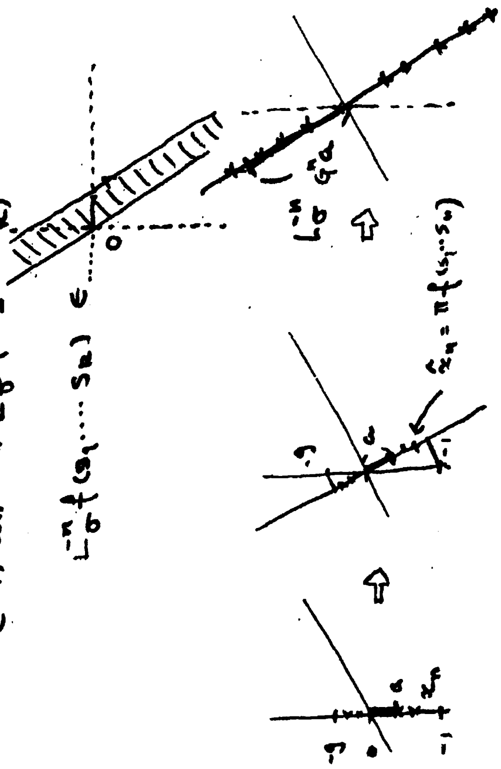
How can we get (2)?

From $\overset{L_\sigma^{-1}}{\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}} \begin{bmatrix} -1 \\ G \end{bmatrix} = -G \begin{bmatrix} -1 \\ G \end{bmatrix}$

$x_n = \hat{\pi} f(s_1 \cdots s_k)$

$(-G)^n x_n = \hat{\pi} L_\sigma^{-n} f(s_1 \cdots s_k)$

$L_\sigma^{-n} f(s_1 \cdots s_k) \in$



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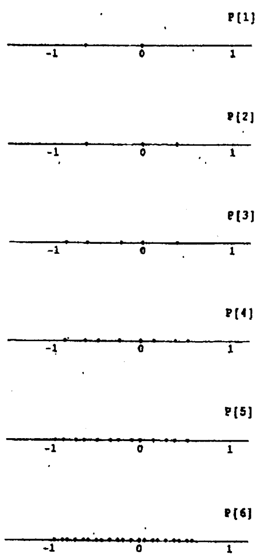


figure of $P_k = \{x_0, x_1, \cdots, x_{g_k - 1}\}$

—12—

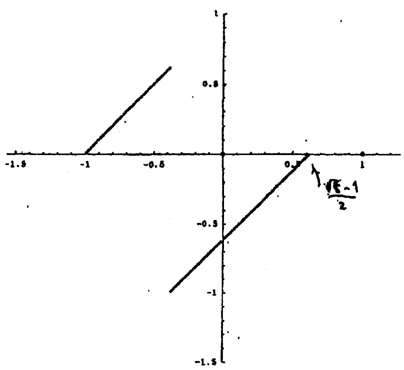


figure of (x_n, x_{n+1})
= graph of T_g

—13—

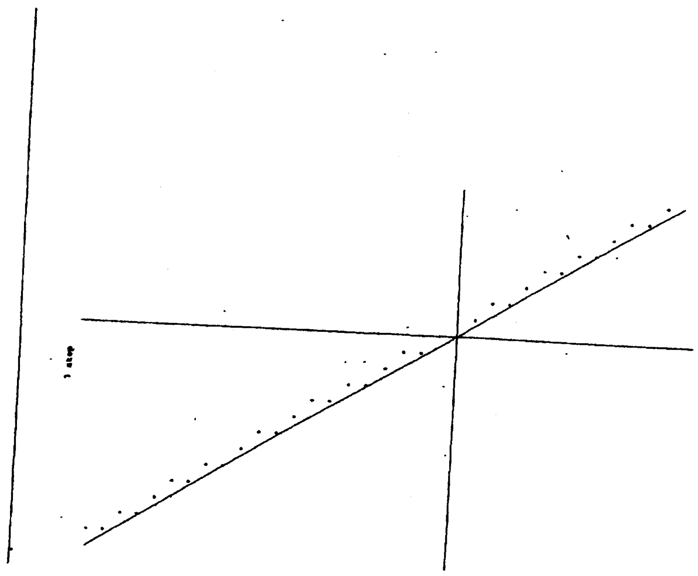
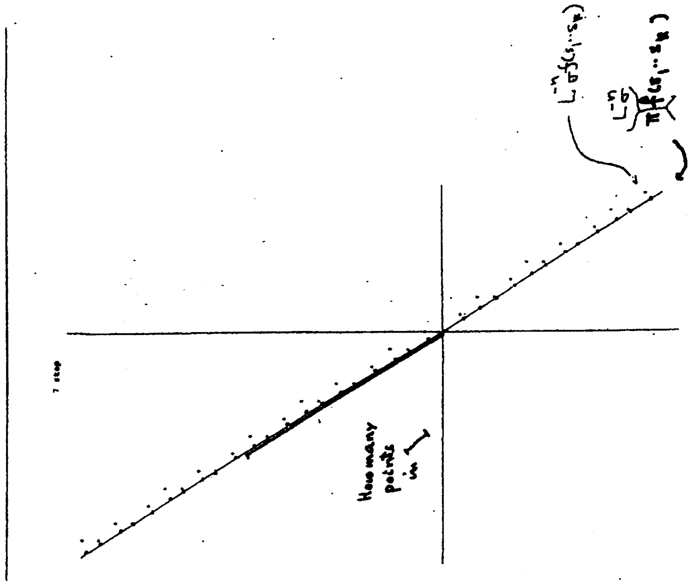


figure of $L_\sigma^{-n} f(s_1 \cdots s_k)$ $0 \le k < a_n$ and $n=7$



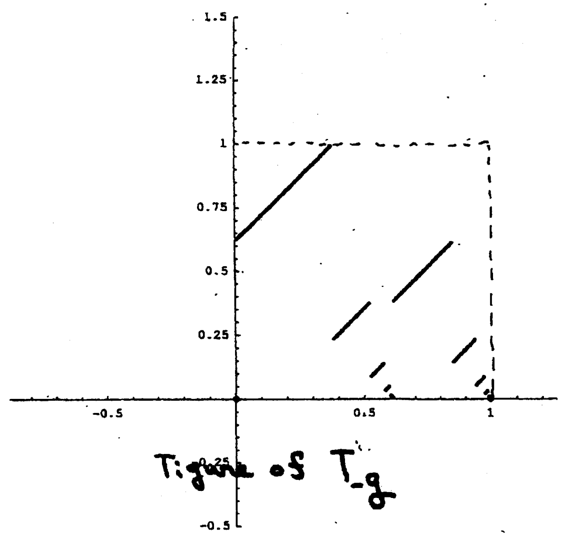


Figure of T_{-g}

Remark 2 For $n = a_k l_k + \cdots + a_0 l_0$

$$\varphi(n) = (a_k, a_{k-1}, \cdots, a_0)$$

$$x_n^* := \sum_{i=0}^{k} a_i (-g)^{i+1} \qquad -g = \frac{\sqrt{5}-1}{2}$$

Then

$$x_n^* = T_{-g}^n(0) \quad , \ n = 0, 1, 2, \cdots$$

(Map T_{-g} should be called the Adding Machine of Fibonacci type.)

Remark 1 $R_\alpha(x) = x + \alpha \pmod 1$

α: irrational

Theorem $x_n := R_\alpha^n(0)$ is a <u>low disc.</u>

$$\boxed{\Longleftrightarrow \quad \frac{1}{m}\sum_{j=1}^{m} a_j \ \text{ is } \underline{\text{bounded}} \qquad \text{where } \alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}}$$

(See Springer Lecture note 1651)

§3 高次元化の試み — (I)

$$X := \left\{ \sum_{j=1}^{\infty} a_j / (-1+i)^{j+1} \;\middle|\; a_j \in \{0,1\} \right\}$$

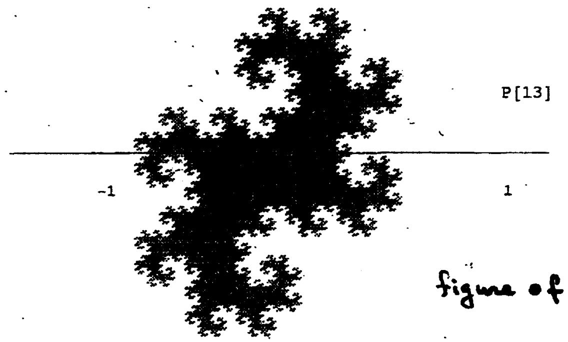


figure of X

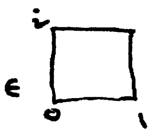
For $n = a_k 2^k + a_{k-1} 2^{k-1} + \cdots + a_0 2^0$

$$\boxed{x_n := \sum_{j=0}^{k} \frac{a_j}{(i-1)^{j+1}} \in} \mathbb{C}$$

we call <u>generalized</u> <u>von der</u> <u>Corput</u> seq.

or

$$\boxed{\hat{x}_n :\equiv x_n \pmod 1} \quad \hat{x}_n \in$$



and $\exists\, T_K : X \to X$ (Kakutani adding M.)

s.t. $\boxed{x_n = T_K^n(0)}$

$P_k := \{x_0, x_1, \cdots, x_{2^n-1}\}$
$\hat{P}_k := \{\hat{x}_0, \cdots, \hat{x}_{2^n-1}\}$

then property

$$\boxed{\begin{array}{l} \text{(i)} \quad P_k = P_{k-1} \cup \left(P_{k-1} - \frac{1}{(i-1)^k}\right) \quad \text{Self similar} \\ \text{In paticular} \\ \widehat{P}_{2m} = \left\{ \frac{p}{2^m} + \frac{q}{2^m} i \;\middle|\; 0 \le q, p < 2^m \right\} \end{array}}$$

See figure

But P_k has not Bounded Remainder.

$$\left| \#\, \widehat{P}_{2m} \cap \underbrace{\boxed{}}_{a}{}^{b} - ab \cdot 2^{2m} \right| < 2^m$$

Therefore $\quad N \doteqdot 2^{2m} \quad 2^m \doteqdot \sqrt{N}$

$$D_N \leqq \sqrt{N} \log N$$

$X(\varepsilon_1, \cdots, \varepsilon_k) := \left\{ \sum_{j=1}^{\infty} a_j / (-1+i)^{j+1} \;\middle|\; (a_1 \cdots a_k) = (\varepsilon_1 \cdots \varepsilon_k) \right\}$

is a bounded remainder set w.r.t. $\{P_k\}$

$\Rightarrow$ Fujita-Ninomiya-Ito.

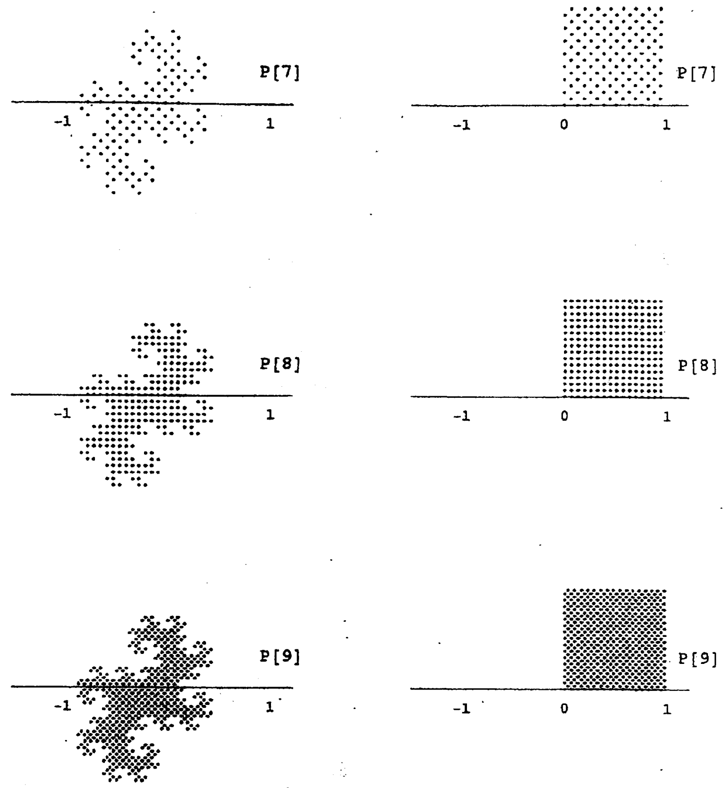


figure of P_k and $\hat{P}_k$, $k = 7, 8, 9$

Halton Sequence

$n = a_k \underline{2}^k + a_{k-1}\underline{2}^{k-1} + \cdots + a_0$

$n = b_j \underline{3}^j + b_{j-1}\underline{3}^{j-1} + \cdots + b_0$

$$\boxed{x_n := \left(\sum_{\ell=0}^{k} a_\ell 2^{-(\ell+1)} , \sum_{\ell=0}^{j} b_\ell 3^{-(\ell+1)} \right)}$$

is called Halton seq.

Theorem On Halton sequence

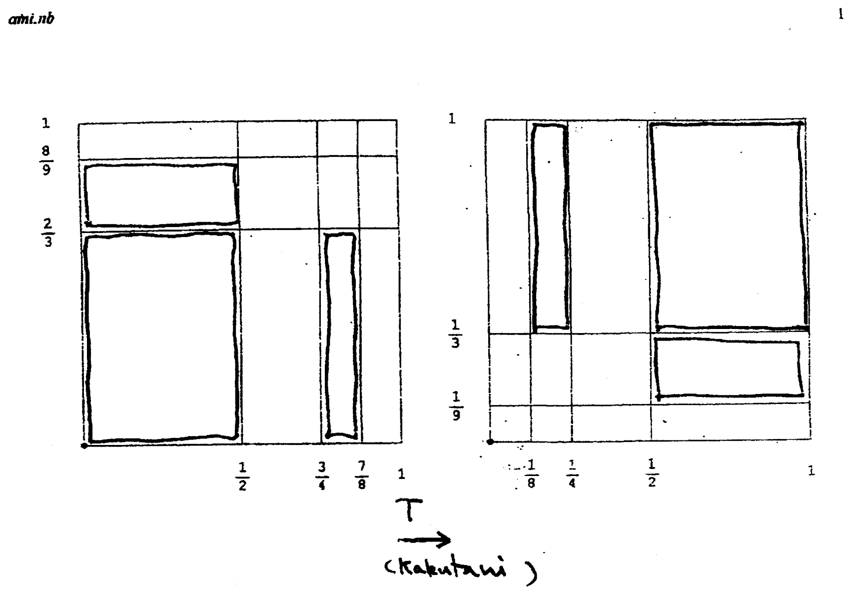
$$\boxed{D_N \sim \log^2 N \cdot N \quad (\text{low discrepancy})}$$

Remark M. Mori construct the 2-dim <u>Dynamical System</u> T_M on $[0,1]^2$

such that

$\underline{x_n = T_M^n(0)}$

and $\{x_n\}$ is <u>low discrepancy</u>.



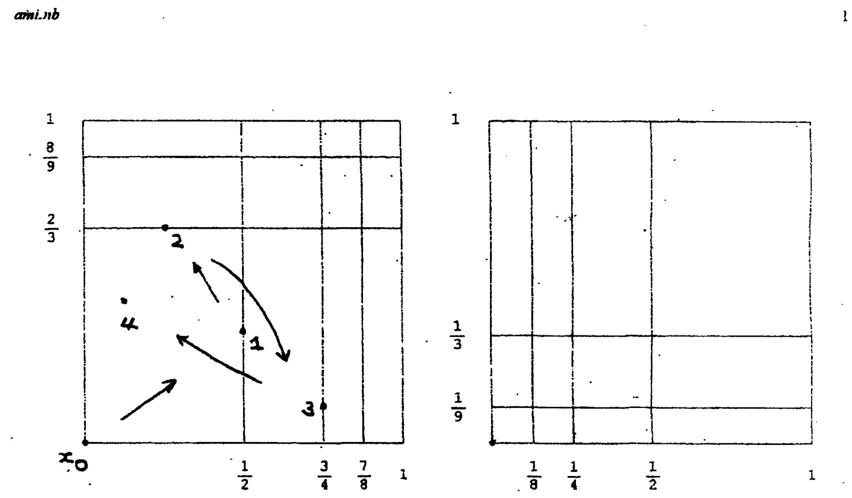
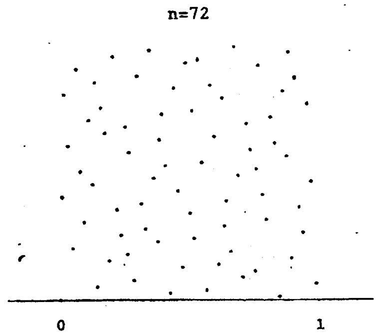
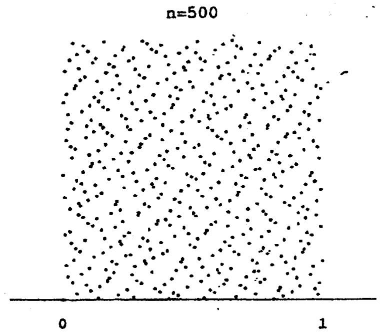


figure of orbit.





Where is
self similar?

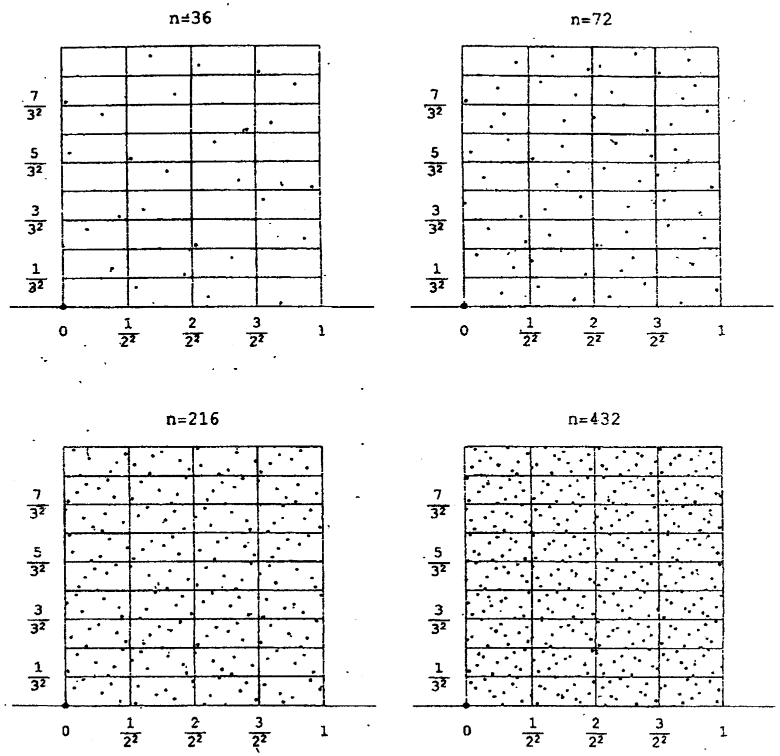


figure of $\{x_k \mid k=0,\cdots,n-1\}$
of <u>Halton</u> Sequence.

Please attension the cardinarity of points
in Box.

Where is $\left\{ \begin{array}{l} \text{self-similar structure ?} \\ \text{bounded remainder property ?} \end{array} \right.$

$$P^{(p,q)} := \{ \bar{x}_n \mid 0 \leq n < 2^p 3^q \}$$

$$E^{(p,q)}_{(c,d)} := \left[\frac{c}{2^p}, \frac{c+1}{2^p} \right) \times \left[\frac{d}{3^q}, \frac{d+1}{3^q} \right)$$

property

(1) $\bar{x}_n \to (c(n), d(n))$, $\bar{x}_n \in E^{(p,q)}_{c(n)d(n)}$

is bijective

$$\left(\#\left\{ \begin{array}{l} 0 \leq n < \cancel{k}\, 2^p 3^q \text{ かつ} \\ \bar{x}_n \in E^{(p,q)}_{c(n),d(n)} \end{array} \right\} = k \right)$$ By Chinese remainder th.

⇨

Property $E^{(p,q)}_{c,d}$ is bounded remainder set

((1)のような "self-similar")

高次元化の試み — (II)

$$\sigma : \begin{array}{l} 1 \to 12 \\ 2 \to 13 \\ 3 \to 1 \end{array} \qquad L_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda > 1 > |\lambda'| = |\lambda''|$$

$|\sigma^n(1)| = l_n$,

$l_n = l_{n-1} + l_{n-2} + l_{n-3}$, $\quad l_0 = 1 \;\; l_1 = 2, \;\; l_2 = 4$

$\sigma^n(1) = s_1 s_2 \cdots s_{l_n}$

$\hat{\pi} : \mathbb{R}^3 \to yz$面 along $\begin{bmatrix} 1 \\ 1/\lambda \\ 1/\lambda^2 \end{bmatrix}$

$$\boxed{x_n := \hat{\pi} f(s_1 \cdots s_n)}$$

by $\quad n = a_k l_k + a_{k-1} l_{k-1} + a_{k-2} l_{k-2} + \cdots + a_0 l_0$

$$x_n = \sum_{i=0} a_i \hat{\pi} \begin{pmatrix} q_i \\ p_i \\ r_i \end{pmatrix} = \sum_{i=0}^{k} a_i \begin{pmatrix} q_i \alpha - p_i \\ q_i \beta - r_i \end{pmatrix}$$

Theorem (Arnoux-I)

$X :=$ closure of $\{x_n\}$

then

(1) $\bigcup_{\hat{\pi} z \in L} (X + \hat{\pi} z) = \mathbb{R}^2$

$L = \{ n(\hat{\pi} e_2 - \hat{\pi} e_1) + m(\hat{\pi} e_3 - \hat{\pi} e_1) \mid n, m \in \mathbb{Z} \}$

, that is,

$$\mathbb{R}^2 / L = X \cong \mathbb{T}^2$$

(2) $T : X \longrightarrow X$

$x \longmapsto x - \hat{\pi} e_1 \pmod{L}$

that is, T is a Rotation on $\mathbb{T}^2$

(3) $x_n = T^n(0)$

$P_k := \{ x_0, x_1, \cdots, x_{q_k - 1} \}$

then

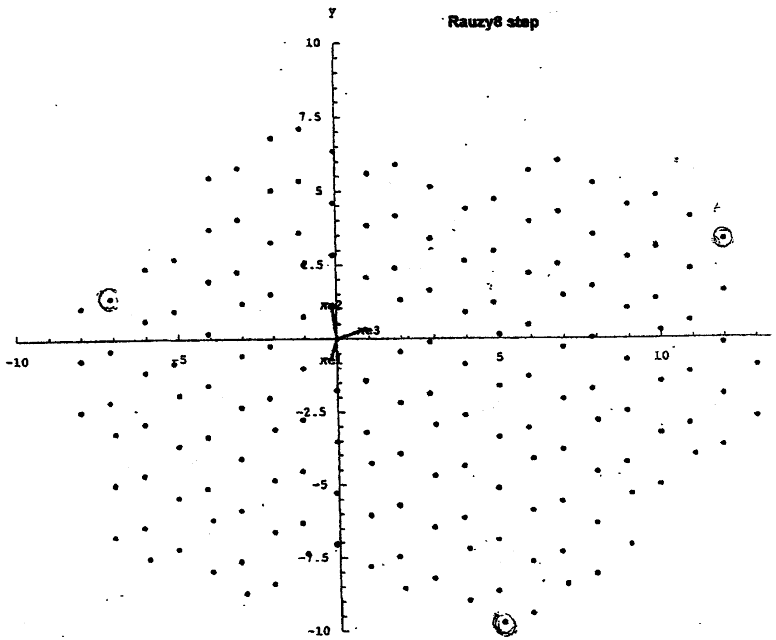
$$\boxed{(1)\ P_k = P_{k-1} \cup \left(P_{k-2} + \hat{\pi}\begin{pmatrix} q_k \\ p_k \\ r_k \end{pmatrix}\right) \cup \left(P_{k-3} + \hat{\pi}\begin{pmatrix} q_{k-1} \\ p_{k-1} \\ r_{k-1} \end{pmatrix}\right)}$$

$\begin{pmatrix} \alpha q_k - p_k \\ \beta q_k - r_k \end{pmatrix}$

self-similarity

For any $n = q_{k_1} + q_{k_2} + \cdots + ~~q_{k_l}~~$

$$P_n = P_{k_1} \cup \left(P_{k_2} + \hat{\pi}\begin{pmatrix} q_{k_1} \\ p_{k_1} \\ r_{k_1} \end{pmatrix}\right) \cup \cdots \cup \left(P_{k_l} + \hat{\pi}\begin{pmatrix} q_{k_{l-1}} \\ p_{k_{l-1}} \\ r_{k_{l-1}} \end{pmatrix}\right)$$



σ: $1 \to 12$, $2 \to 13$, $3 \to 1$

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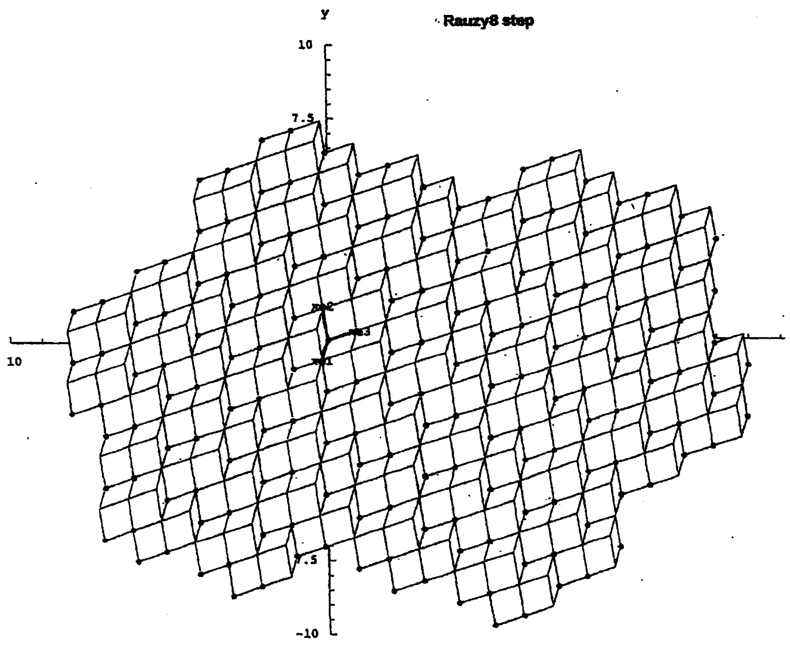
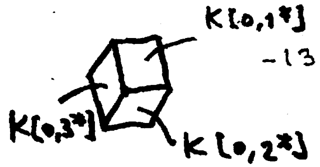
Signe of P_k, $k = 8$

$P_k = P_{k-1} + \bigcirc + \bigcirc$





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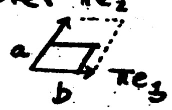
— 26-1/2 —

Bounded Remainder

let $\square(a,b) :=$ 

$$A_k^{(a,b)} = \left| \#\left(P_k \cap \square(a,b) \right) - |\square(a,b)| \cdot l_k \right|$$

if we find c :

$$A_k(a,b) < c\,k$$

then $\left| \#\{x_n \mid n = 0, \cdots, N-1\} \cap \square(a,b) - |\square(a,b)| \cdot N \right|$

$$\leqq c' k^2 \doteqdot c \log^2 N$$

Lemma

$$\#\left(P_k \cap \square(a,b)\right) = \#\left(\pi S \cap L_\sigma^{-k} \square(a,b)\right)$$

where $\pi S = \{\pi x \mid x \in \mathbb{Z}^3, (x, v) \geq 0, (x - e_1, v) < 0\}$

Question

$$\#\, \pi S \cap L_\sigma^{-k} \square(a,b) = ?$$

Remark ⇨ Fujita-Ito

$$\left(X_{(\varepsilon_1 \cdots \varepsilon_k)} := \left\{ \sum_{i=1}^{\infty} a_i \lambda'^{\,i} \;\middle|\; (a_1 a_2 \cdots a_k) = (\varepsilon_1 \cdots \varepsilon_k) \right\} \right.$$

is a bounded remainder set w.r.t. $\{P_k\}$.)

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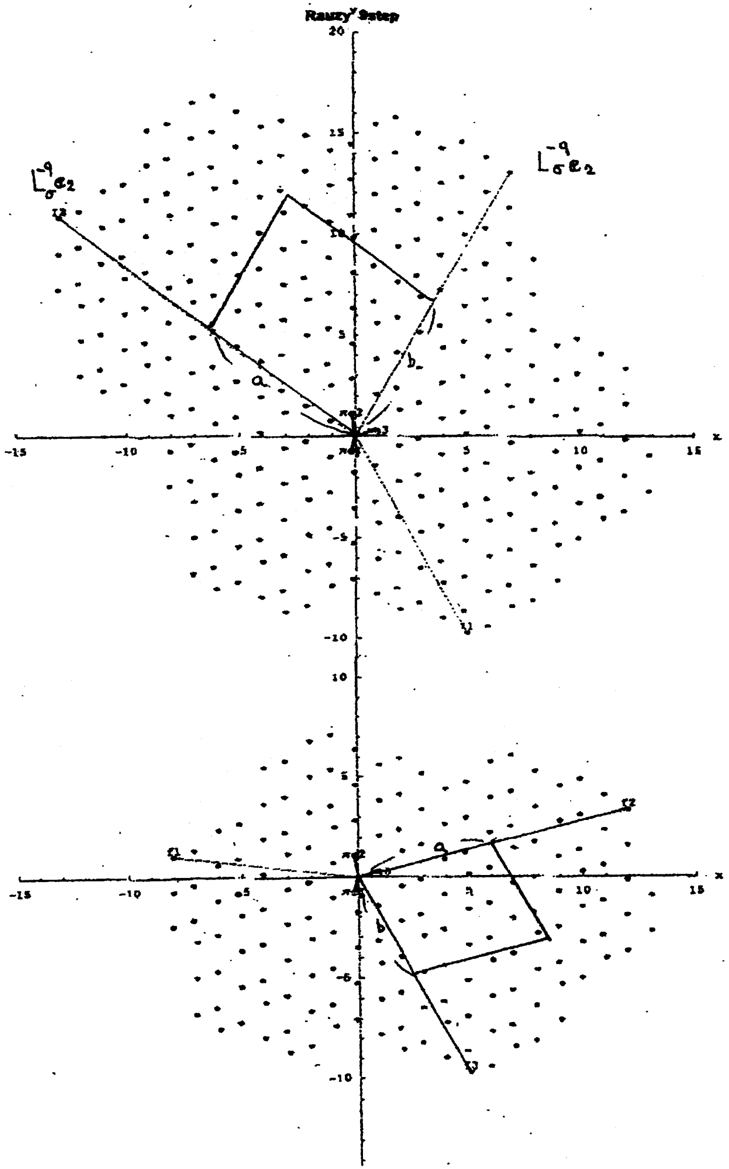

$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s)$

$1, \alpha_1, \alpha_2, \cdots, \alpha_s$: linearly ind over $\mathbb{Q}$

$x_n := (\{n\alpha_1\}, \ldots, \{n\alpha_s\})$

<u>Remark</u>

- $D_N(\{x_n\}) = O(N^{-1}(1 + \log N)^{s+1})$

- α: algebraic

  $D_N(\{x_n\}) = O(N^{-1+\varepsilon})$ for every $\varepsilon > 0$

  (Niederreiter)

- Assume $\exists \eta \geq 1$, $c > 0$:

  $\underline{r^{\eta}((h_1, \cdots, h_s)) \langle h_1\alpha_1 + \cdots + h_s\alpha_s \rangle \geq c}$

  for all $(h_1, \cdots, h_s) \neq (0, \cdots, 0) \in \mathbb{Z}^s$

where $r(h) := \prod_{j=1}^{s} \max(|h_j|, 1)$

then

$D_N(\{x_n\}) = O(N^{-1} \log^{s+1} N)$ <u>if $\eta = 1$</u>

$D_N(\{x_n\}) = O(N^{-1/((\eta-1)s+1)} \log N)$

if $\eta > 1$

proof is obtained from
Erdös-Turán-Koksma's ineq.